Quantum Algorithms and Sparse Optimization

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1 Background

Recent advances in quantum computing have enabled novel approaches to solving optimization problems, including quantum-enhanced linear programming and semidefinite programming solvers. This project focuses on developing specialized quantum algorithms for sparse optimization problems, exploring both theoretical foundations and practical implementations.

1.1 Sparse Convex Optimization Problems

Generally, we consider constrained convex optimization problems of the following form

$$\min_{x \in \mathcal{D}} f(x),$$

where f is a convex and L-smooth function (L-smooth means the gradient of f is L-Lipschitz).

The Frank-Wolfe method (also known as the conditional gradient method) is a widely used way to solve this kind of problems. It's like performing a gradient descent with respect to some constraint set. See https://people.csail.mit.edu/stefje/fall15/notes_lecture14.pdf for more details about this method. In short, Frank-Wolfe method works as follows:

Algorithm 1 Frank-Wolfe Optimization Algorithm

Require: Domain $\mathcal{D}$, function f, number of iterations T

Initialize $w^0 \in \mathcal{D}$
for $t = 1, 2, \ldots, T$ **do**
  Compute $s^t \in \arg\max_{s \in \mathcal{D}} \langle s, \nabla f(w^t) \rangle$
  Set step size $\eta_t \leftarrow \frac{1}{t+2}$
  $w^{t+1} \leftarrow (1 - \eta_t) w^t + \eta_t s^t$
end for
return w^T

2 Problems: Quantum Frank-Wolfe

In this part, we will explore quantum implementations of the Frank-Wolfe method.

2.1 Basic Questions

We consider the simple case where $\mathcal{D}$ is a d-dimensional probability simplex Δ_d, i.e.,

$$\mathcal{D} = \Delta_d := \left\{ x \in \mathbb{R}^d \mid x_i \geq 0, \sum_i x_i = 1 \right\},$$

and f is a quadratic function $f(x) := \|Ax + b\|^2$, where $A \in \mathbb{R}^{n\times d}$ and $b \in \mathbb{R}^n$. Suppose we are given quantum query access to A and b.

The question is to give a quantum implementation of the Frank-Wolfe method with complexity that has $O(\sqrt{d})$ dependence.

2.2 Advanced Questions

Here are some potential questions that you could work with.

2.2.1 Changing the norm constraint

Suppose f is a quadratic function $f(x) := \|Ax + b\|^2$, where $A \in \mathbb{R}^{n\times d}$ and $b \in \mathbb{R}^n$. Suppose we are given quantum query access to A and b. What is the optimal quantum algorithm when $\mathcal{D}$ is a $\|\cdot\|_p$-ball? Discuss the cases for $p = 2, \infty$.

2.2.2 Generalizing to matrix functions

Consider $\mathcal{D} = \{X \in \mathbb{R}^{d\times d} \mid X \succeq 0, \operatorname{tr}(X) = 1\}$, and quantum query access to ∇f is given.

The question is to give a quantum implementation of the Frank-Wolfe method with good complexity guarantees.

You could also discuss various settings studied in http://proceedings.mlr.press/v28/jaggi13-supp.pdf.

3 Useful Materials

Here are some quantum algorithm techniques that may be useful.

- You may find https://quantumalgorithms.org useful, especially some quantum linear algebra tools such as block-encodings and quantum singular value transformation.

- KP-trees. KP-tree is a quantum implementation of the segment tree that cooperates well into the state preparation task. See https://arxiv.org/abs/1603.08675.
- Quantum minimum finding and generalized minimum finding. See https://arxiv.org/abs/1705.01843 Appendix C for more details on the latter one.

For analysis of the Frank-Wolfe method, see http://proceedings.mlr.press/v28/jaggi13-supp.pdf for more details.

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